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DYNAMIC BUCKLING OF VISCOPLASTIC SPHERICAL SHELLS

Robert Lincoln Dickey

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DYNAMIC BUCKLING OF
VISCOPLASTIC SPHERICAL SHELLS

by

LIEUTENANT COMMANDER ROBERT LINCOLN DICKEY, USN
B.C.E. Rensselaer Polytechnic Institute
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Dynamic Buckling of Viscoplastic Spherical Shells

by

Lieutenant Commander Robert Lincoln Dickey, USN

Submitted to the Department of Ocean Engineering on May 10, 1074, in partial fulfillment of the requirements for the Degree of Ocean Engineer and for the Degree of Master of Science in Naval Architecture and Marine Engineering.

Abstract

Buckling is a significant cause of structural failure. A theoretical investigation has been undertaken into the dynamic instability of complete spherical shells which are loaded impulsively and made from a viscoplastic material.

The critical mode numbers obtained in this thesis are similar to those obtained for cylindrical shells having the same R/h ratios and material parameters while the threshold velocities were for higher viscoplastic spherical shells than for rigid-plastic spherical shells.

Thesis Supervisor: Norman Jones

Title: Associate Professor of Ocean Engineering

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List of Symbols

a_{mn}, b_{mn}	non-dimensional displacement amplitude (15a)
a_n, b_n	variable defined in (37e, 37f)
c_{mn}, d_{mn}	non-dimensional stress function (15b)
$\dot{e}_{ij}$	membrane strain rate
h	thickness of shell
i,j,m,n	indices
k	yield stress in simple shear $\sigma_o/\sqrt{3}$
$t(t_f)$	time variable (final)
u	non-dimensional perturbation displacement W'/R
u_o	non-dimensional dominant displacement $\overline{W}/R$
z	normal coordinate
A_{mn}	displacement amplification function
B_{mn}	velocity amplification function
C	equivalent velocity
F	stress function
$\tilde{F}$	non-dimentional stress function
J_2	second invarient of the stress deviator
K	constant of integration
M_{ij}	bending or twisting moment
$M(\ )$	Kummer function of first kind
N_{ij}	resultant membrane stress
P	static pressure
$\tilde{P}$	non-dimensional pressure
$Q_n^{(1)}$, $Q_n^{(2)}$	variables defined (34b, 34c)

R	radius of shell
R_n^2	variable defined (34d)
S_{ij}	deviation stress
U, V, W	displacements
$U(\ )$	Kummer function of second kind
V_o	initial velocity
α^2	$1/12\ (h/R)^2$
α_n	non-dimensional radial shell imperfections
β_n	non-dimensional velocity imperfections
β^2	$C/(R\gamma^*)$
γ	viscosity coefficient
$\gamma_1,\ \gamma^*$	viscosity variables defined (2.5, 1.6d)
Γ	gamma function
δ	Cowper and Symonds power constant
$\dot{\varepsilon}_e(\dot{\varepsilon}_e^*)$	equivalent strain rate (initial)
$\dot{\varepsilon}_{ij}$	plastic strain rate tensor
θ	circumferential coordinates (angles)
$\dot{\kappa}_{ij}$	curvature change rates
λ_n	$n(n + 1)$
Λ_n	variable defined (37h)
μ	slope of stress-strain rate (2.2)
ξ	time variable $1 - \tau/\tau_f$
ρ	density of materials
σ_{ij}	stress tensor
$\sigma_o(\sigma_e,\ \sigma_{01})$	yield (equivalent, linearized) stress
$\tilde{\sigma}$	non-dimensional stress

$\tau(\tau_f)$	non-dimensional time (terminal)
ϕ	meridional coordinate
$\tilde{\omega}_n$	variable defined (16b)
Ω_n	variable defined (37g)
$(\dot{\ })$	$\frac{\partial}{\partial t}(\ )$
$(\ )_{,\tau}$	$\frac{\partial}{\partial \tau}(\ )$
$\nabla^2(\ )$	defined (11f)

Introduction

The analysis of dynamic buckling, of elastic-plastic and rigid plastic cylinders has been investigated by Abrahamson and Goodier [1] extended by Goodier [2], Goodier and Florence [3] and others [4-6]. Florence [7-8] and Wojewodzki [9-11] extended the analysis to viscoplastic cylindrical shells. Work on spherical shells has progressed through elastic-plastic and rigid plastic materials [12-14]. The analytical procedures of the above references were derived by assuming that buckling stemmed from the growth of small imperfections in otherwise uniform initial displacements and velocity fields.

The dynamic buckling of a thin-walled complete spherical shell which is made from a viscoplastic material is investigated herein. A theoretical procedure is developed by first considering the unperturbed motion arising from a spherically symmetric initial velocity field. An examination is then made of the influence of small perturbations in this otherwise uniform velocity field and of the effect of small deivations from the initial sphericity of the shell. It transpires that certain harmonics grow very quickly and cause a spherical shell to exhibit a characteristic wrinkled shape. It is assumed that the strains which are associated with the perturbed state are much smaller than those associated with the predominant motion. Thus, no unloading of the shell material occurs before the time when all the

initial kinetic energy is absorbed by the spherical shell in the predominant mode. Moreover, it is further assumed that the wrinkled pattern in the shells is established prior to this time and is not modified by subsequent unloading.

Chapter 2

Basic Equations

The equilibrium equations, compatibility relations and strain and curvature relations have been presented in refs. [13] and [14] for a spherical shell. These results are repeated in this section for convenience.

It may be shown for the particular case of a thin-walled spherical shell that the strain and curvature relations which were developed in reference [12] for an arbitrarily shaped shell, can be written

$$\dot{e}_{\phi} = \frac{1}{R}\left(U_{,\phi} - \dot{W} + \frac{1}{R} W_{,\phi}\dot{W}_{,\phi}\right) \tag{1a}$$

$$\dot{e}_{\theta} = \frac{1}{R}\left(\mathrm{cosec}\phi\ \dot{V}_{,\theta} + \cot\phi\ \dot{U} - \dot{W} + \frac{1}{R}\mathrm{cosec}^2\phi\ W_{,\theta}\dot{W}_{,\theta}\right) \tag{1b}$$

$$\dot{e}_{\phi\theta} = \frac{1}{R}\Big[\mathrm{cosec}\phi\ \dot{U}_{,\theta} - \cot\phi\ \dot{V} + \dot{V}_{,\phi} + \frac{1}{R}\mathrm{cosec}\phi\ (W_{,\phi}\dot{W}_{,\theta} + W_{,\theta}\dot{W}_{,\phi})\Big] \tag{1c}$$

$$\dot{\kappa}_{\phi} = \frac{1}{R^2}\left(\dot{W} + \dot{W}_{,\phi\phi}\right) \tag{1d}$$

$$\dot{\kappa}_{\theta} = \frac{1}{R^2}\left(\dot{W} + \mathrm{cosec}^2\phi\ \dot{W}_{,\theta\theta} + \cot\phi\ \dot{W}_{,\phi}\right) \tag{1e}$$

and

$$\dot{\kappa}_{\phi\theta} = \frac{2}{R^2}\left[\mathrm{cosec}\phi\ \dot{W}_{,\phi\theta} - \cot\phi\ \mathrm{cosec}\phi\ \dot{W}_{,\theta}\right] \tag{1f}$$

when the displacements are defined as shown in Figure 1. A straightforward modification of the general equations in ref. [12] for a shell of thickness h and density ρ gives the equilibrium equations

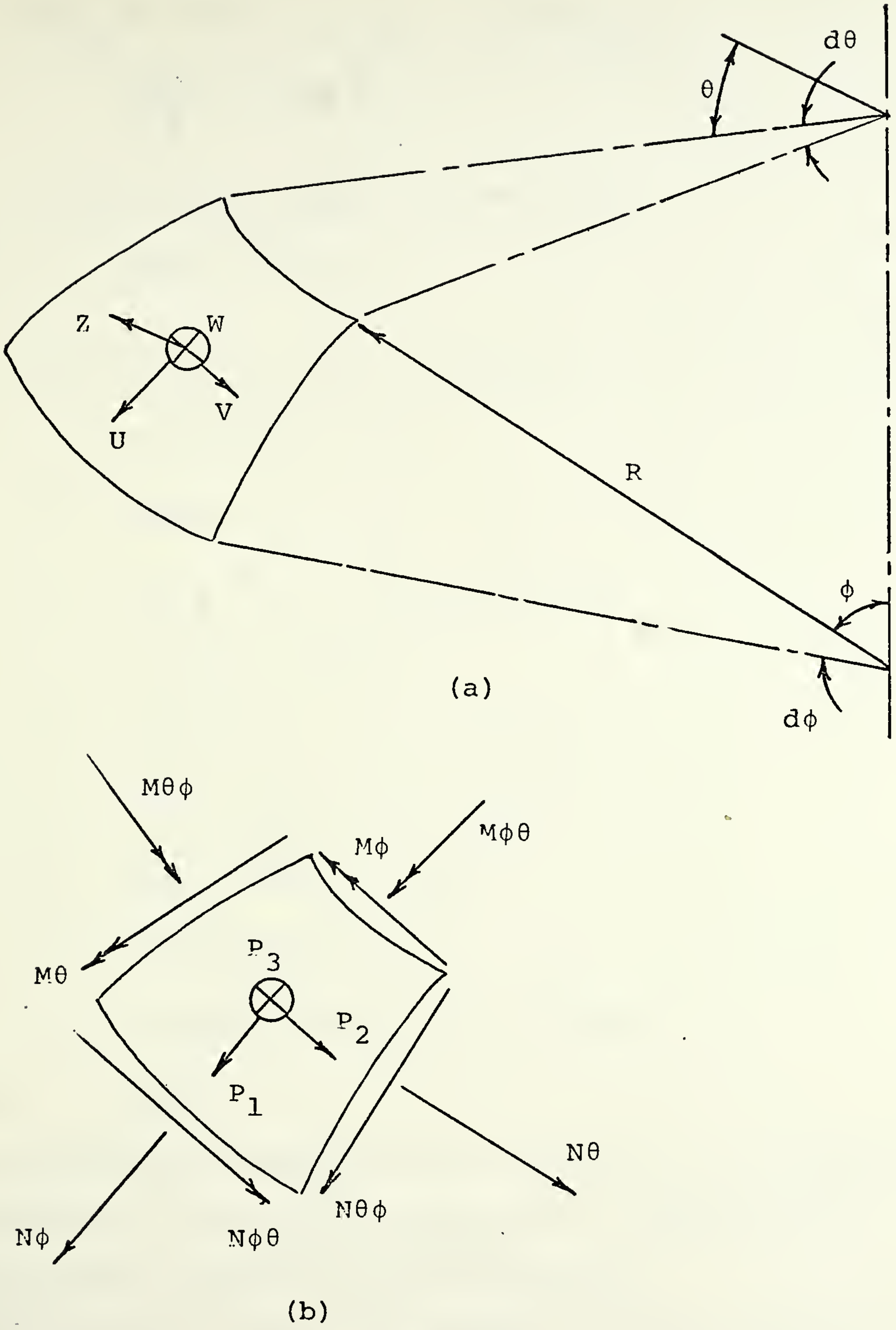


Figure 1: (a) Displacement and (b) Membrane Forces, Bending Moments and External Loads on a Spherical Shell.

$$\begin{aligned}&N_{\phi,\phi} + \cot\phi\ (N_\phi - N_\theta) + \operatorname{cosec}\phi\ N_{\phi\theta,\theta} - \\ &\qquad \rho h R\ddot{U} + P_1 R = 0\end{aligned} \tag{2a}$$

$$\begin{aligned}&\operatorname{cosec}\phi\ N_{\theta,\theta} + N_{\phi\theta,\phi} + 2\cot\phi\ N_{\phi\theta} - \\ &\qquad \rho h R\ddot{V} + P_2 R = 0\end{aligned} \tag{2b}$$

and

$$\begin{aligned}&R(N_\phi + N_\theta) + N_\phi(W_{,\phi\phi} + \cot\phi\ W_{,\phi}) + \\ &\qquad N_\theta\ \operatorname{cosec}^2\phi\ W_{,\theta\theta} + \\ &\qquad \operatorname{cosec}\phi\ N_{\phi\theta}\ (W_{,\phi\theta} + W_{,\theta\phi}) + \\ &\qquad (N_{\phi,\phi}\ W_{,\phi} + W_{,\theta}N_{\phi\theta,\phi}\ \operatorname{cosec}\phi + \\ &\qquad W_{,\phi}\ N_{\phi\theta,\theta}\ \operatorname{cosec}\phi + \\ &\qquad \operatorname{cosec}^2\phi\ N_{\theta,\theta}\ W_{,\theta}) - (M_{\phi,\phi\phi} + \\ &\qquad 2\cot\phi\ M_{\phi,\phi} - \cot\phi\ M_{\theta,\phi} + \\ &\qquad 2M_\theta + \operatorname{cosec}^2\phi\ M_{\theta,\theta\theta} + \\ &\qquad 2\operatorname{cosec}\phi\ M_{\phi\theta,\phi\theta} + \\ &\qquad 2\operatorname{cosec}\phi\ \cot\phi\ M_{\phi\theta,\theta}) - R^2\rho h\ddot{W} + \\ &\qquad R^2 P_3 = 0\end{aligned} \tag{2c}$$

which are consistent with equations (1a) and (1f) according to the principle of virtual work provided the influence of transverse shear deformation is disregarded.

The strain and change of curvature relations (1a) to (1f) satisfy the compatibility equation

$$\begin{aligned}&\dot{e}_{\theta,\phi\phi} + \text{cosec}^2\phi\ \dot{e}_{\phi,\theta\theta} + 2\ \cot\phi\ \dot{e}_{\theta,\phi} - \\ &\quad \cot\phi\ \dot{e}_{\phi,\phi} + 2\dot{e}_{\phi} - \text{cosec}\phi\ \dot{e}_{\phi\theta,\phi\theta} - \\ &\quad \text{cosec}\phi\ \cot\phi\ \dot{e}_{\phi\theta,\theta} + R(\dot{\kappa}_{\phi} + \dot{\kappa}_{\theta}) = 0 \qquad (3)\end{aligned}$$

for infinitesimal displacements [12].

The constitutive equations for spherical shells which are made from rigid viscoplastic material are developed in Appendix A. Equations (1.8a) to (1.8f) for a spherical shell made from rigid viscoplastic material can be written in the form

$$N_{\phi} = \frac{kh}{\gamma^*}(2\dot{e}_{\phi} + \dot{e}_{\theta}) \qquad (4a)$$

$$N_{\theta} = \frac{kh}{\gamma^*}(2\dot{e}_{\theta} + \dot{e}_{\phi}) \qquad (4b)$$

$$N_{\phi\theta} = \frac{kh}{\gamma^*}\,\frac{\dot{e}_{\phi\theta}}{2} \qquad (4c)$$

$$M_{\phi} = \frac{kh^3}{12\gamma^*}(2\dot{\kappa}_{\phi} + \dot{\kappa}_{\theta}) \qquad (4d)$$

$$M_{\theta} = \frac{kh^3}{12\gamma^*}(2\dot{\kappa}_{\theta} + \dot{\kappa}_{\phi}) \qquad (4e)$$

and $$M_{\phi\theta} = \frac{kh^3}{12\gamma^*}\,\frac{\dot{\kappa}_{\phi\theta}}{2} \qquad (4f)$$

where $$k = \sigma_o/\sqrt{3} \qquad (5a)$$

and $$\gamma^* = \gamma(1 - k/\sqrt{J_2}). \qquad (5b)$$

The method of solution used here is similar to that used in refs. [12] and [13]. It is assumed that the displacement in a complete spherical shell consists of dominant parts

$(\overline{W},\ \overline{U} = \overline{V} = 0)$ and perturbation parts $(w',\ U',\ V')$ so that

$$W(\phi,\theta,\tau) = \overline{W}(\tau) + W'(\phi,\theta,\tau) \tag{6a}$$

$$U(\phi,\theta,\tau) = U'(\phi,\theta,\tau) \tag{6b}$$

and

$$V(\phi,\theta,\tau) = V'(\phi,\theta,\tau). \tag{6c}$$

Thus, the dominant strains and curvature changes are

$$\overline{e}_\phi = \overline{e}_\theta = -\frac{\overline{W}}{R}, \quad \overline{e}_{\phi\theta} = 0. \tag{7a-c}$$

and

$$\overline{\kappa}_\phi = \overline{\kappa}_\theta = \frac{\overline{W}}{R^2}, \quad \overline{\kappa}_{\phi\theta} = 0 \tag{7d-f}$$

respectively, while the corresponding perturbation quantities are given by equation (1a) to (1f) with primes. Similarly,

$$N_\phi(\phi,\theta,\tau) = \overline{N}(\tau) + N'_\phi(\phi,\theta,\tau) \tag{8a}$$

$$N_\theta(\phi,\theta,\tau) = \overline{N}(\tau) + N'_\theta(\phi,\theta,\tau) \tag{8b}$$

$$N_{\phi\theta}(\phi,\theta,\tau) = N'_{\phi\theta}(\phi,\theta,\tau) \tag{8c}$$

$$M_\phi(\phi,\theta,\tau) = \overline{M}(\tau) + M'_\phi(\phi,\theta,\tau) \tag{8d}$$

$$M_\theta(\phi,\theta,\tau) = \overline{M}(\tau) + M'_\theta(\phi,\theta,\tau) \tag{8e}$$

and

$$M_{\phi\theta}(\phi,\theta,\tau) = M'_{\phi\theta}(\phi,\theta,\tau). \tag{8f}$$

If equations (6) and (8) are substituted into equation (2c), then the dominant and perturbation transverse equilibrium equations are

$$2\,\frac{\overline{N}}{R} - 2\,\frac{\overline{M}}{R^2} + P_3 - \rho h\ddot{\overline{W}} = 0 \tag{9a}$$

and $$R(N'_\phi + N'_\theta) + \overline{N}(W'_{,\phi\phi} + \cot\phi\ W'_{,\phi} + \mathrm{cosec}^2\phi\ W'_{,\theta\theta}) -$$
$$[M'_{\phi,\phi\phi} + 2\cot\phi\ M'_{\phi,\phi} - \cot\phi\ M'_{\theta,\phi} +$$
$$2M'_\theta + \mathrm{cosec}^2\phi\ M'_{\theta,\theta\theta} + 2\ \mathrm{cosec}\phi\ M'_{\phi\theta,\theta\phi} +$$
$$2\ \mathrm{cosec}\phi\ \cot\phi\ M'_{\phi\theta,\theta}] - \rho h R^2 \ddot{W}' = 0 \qquad (9b)$$

respectively, when neglecting higher order quantities.

It may be shown that in-plane equilibrium equations (2a) and (2b) are satisfied identically when

$$N_\phi = \frac{1}{R^2}[F + \mathrm{cosec}^2\phi\ F_{,\phi\phi} + \cot\phi\ F_{,\phi}] \qquad (10a)$$

$$N_\theta = \frac{1}{R^2}[F + F_{,\phi\phi}] \qquad (10b)$$

and $$N_{\phi\theta} = -\frac{1}{R^2}\mathrm{cosec}\phi\ [F_{,\phi\theta} - \cot\phi\ F_{,\theta}] \qquad (10c)$$

where F is a stress function and $U = V = P_1 = P_2 = 0$. Now substituting the inversions of equations (4a) to (4c) and equations (12a) to (12c) and (1d) to (1e) into the compatibility equation (3) gives

$$3(\nabla^2 + 2)u_{,\tau} + [2(\nabla^2 + 1) - 1](\nabla^2 + 2)\tilde{F} = 0 \qquad (11a)$$

where $$u = \frac{W'}{R}, \quad \tilde{F} = \frac{F\gamma^*}{khRC}, \quad C^2 = \frac{k}{\rho}, \quad \tau = \frac{C}{R}t \qquad (11b\text{-}e)$$

and $$\nabla^2(\ ) = (\ )_{,\phi\phi} + \cot\phi(\ )_{,\phi} + \mathrm{cosec}^2\phi(\ )_{,\theta\theta}. \qquad (11f)$$

The transverse equilibrium equations (9a) and (9b) can be written in the form

$$-u_{o,\tau\tau} + 2(1 + \alpha^2)\tilde{\sigma} + \tilde{P} = 0 \qquad (12a)$$

and

$$u_{,\tau\tau} - \beta^2(\nabla^2 + 2)\tilde{F} - \tilde{\sigma}\nabla^2 u +$$

$$\alpha^2\beta^2[2(\nabla^2 + 1) + 1](\nabla^2 + 2)u_{,\tau} = 0 \qquad (12b)$$

respectively, when using equations (1), (4), (6)-(8) and (10) and where

$$u_o = \frac{\overline{W}}{R}, \quad \tilde{\sigma} = \frac{\overline{N}}{kh}, \quad \tilde{P} = \frac{P_3 R}{kh}, \quad \alpha^2 = \frac{1}{12}\left(\frac{h}{R}\right)^2$$

and

$$\beta^2 = \frac{C}{R\gamma^*}. \qquad (12c\text{-}g)$$

Chapter 3

Static Behavior

Equation (12a) predicts

$$\tilde{\sigma} = -\frac{\tilde{P}}{2} \tag{13}$$

when $\alpha^2 << 1$, while equations (11a) and (12b) become

$$[2(\lambda_n - 1) + 1](\lambda_n - 2)c_{mn} - 3(\lambda_n - 2)a_{mn},_\tau = 0 \tag{14a}$$

and

$$\beta^2(\lambda_n - 2)c_{mn} + \tilde{\sigma}\lambda_n a_{mn} + \alpha^2\beta^2[2(\lambda_n - 1) -1](\lambda_n - 2)a_{mn},_\tau = 0 \tag{14b}$$

respectively, if it is assumed that

$$u = a_{on} P_n(\cos\phi) + P_n^m(\cos\phi)(a_{mn} \cos m\theta + b_{mn} \sin m\theta) \tag{15a}$$

and

$$\tilde{F} = c_{on} P_n(\cos\phi) + P_n^m(\cos\phi)(c_{mn} \cos m\theta + d_{mn} \sin m\theta) \tag{15b}$$

where

$$\lambda_n = n(n + 1) \tag{15c}$$

and P_n and P_n^m are Legendre and associated Legendre polynomials of degree n and order m. Two additional equations are also obtained but these are identical to equations (14a) and (14b) provide b_{mn} and d_{mn} are substituted for a_{mn} and c_{mn}. If c_{mn} is eliminated from equations (14a) and (14b), then

$$\beta^2 \tilde{\omega}_n \, a_{mn,\tau} + \tilde{\sigma} \lambda_n \, a_{mn} = 0 \tag{16a}$$

where $$\tilde{\omega}_n = \frac{3(\lambda_n - 2)}{2\lambda_n - 1} + \alpha^2 (2\lambda_n - 3)(\lambda_n - 2). \tag{16b}$$

Now, substituting equation (13) into (16a) and integrating gives

$$a_{mn} = e^{\left(\frac{\tilde{P}}{2} \frac{\lambda_n}{\tilde{\omega}_n} \frac{1}{\beta^2} \frac{C}{R} t\right)} + K \tag{17}$$

where K is a constant of integration which is related to the initial imperfections in a shell. β^2 which equals $\frac{C}{R\gamma^*}$ depends on strain rate as γ^* depends on strain rate as shown in appendix B. With a strain rate near zero, as would be the case for static buckling β^2 goes to infinity but $1/\beta^2$ becomes

$$\frac{1}{\beta^2} = \frac{R\gamma^*}{C} = \frac{R}{C} \left(\frac{\dot{\varepsilon}_e \gamma}{\dot{\varepsilon}_e + \gamma} \right). \tag{18}$$

In the limit of $\dot{\varepsilon}_e$ very small, $\dot{\varepsilon}_e << \gamma$

$$\lim_{\dot{\varepsilon}_e \to 0} \left(\frac{1}{\beta^2} \right) = \frac{R}{C} \dot{\varepsilon}_e \tag{19}$$

where $\dot{\varepsilon}_e$ is the initial strain rate. Therefore equation (17) becomes

$$a_{mn} = e^{\left(\frac{\tilde{P}}{2} \frac{\lambda_n}{\tilde{\omega}_n} \dot{\varepsilon}_e t\right)} + K \tag{20}$$

when $\dot{\varepsilon}_e << \gamma$.

This equation is of the form of creep buckling.

Chapter 4

Dynamic Buckling

4.1 Dominant Equation

From equation (12a)

$$u_{o,\tau\tau} = 2\tilde{\sigma} \tag{21}$$

when $\alpha^2 << 1$ and $\tilde{P} = 0$. $\tilde{\sigma}$ is not time invariant and the solution of the above equation will be done in two parts. The first part will assume the value of $\tilde{\sigma}$ to be constant with time and the second will give the exact solution, then the two solutions will be equated at the final displacements. For buckling the dominant stress is defined as

$$\bar{N} = - h\sigma_e \tag{22}$$

and equation (21) becomes

$$u_{o,\tau\tau} = - 2\sqrt{3}\,\frac{\sigma_e}{\sigma_o}. \tag{23}$$

The solution of equation (23) is

$$u_o = \frac{V_o}{C}\,\tau\left[1 - \frac{\tau}{2\tau_f}\right] \tag{24a}$$

when satisfying the initial conditions W = 0 and $\ddot{W} = V_o$ at t = 0 and where

$$\tau_f = \frac{V_o}{2\sqrt{3}C}\left(\frac{\sigma_o}{\sigma_e}\right) \tag{24b}$$

is the time when $\dot{W} = 0$.

The exact solution uses equation (4a), (4b), (7a) and (7b) to revise $\tilde{\sigma}$ to reflect the time dependence as follows

$$\bar{N} = \frac{kh}{\gamma^*} \; 3 \left(- \frac{\dot{W}}{R} \right). \tag{25}$$

Solving for $\tilde{\sigma}$ and using equations (12c) and (2.9) yields

$$\tilde{\sigma} = \frac{\bar{N}}{kh} = - \frac{3}{2} - 3 \frac{C}{R\gamma} u_{o',\tau} \tag{26}$$

where

$$\dot{\varepsilon}_e = 2 \frac{C}{R} u_{o',\tau}. \tag{27}$$

Substituting equation (26) into (21) produces

$$u_{o',\tau\tau} + \frac{6C}{R\gamma} u_{o',\tau} = - 3 \tag{28}$$

the equation for the exact solution. The solution of equation (28) is

$$u_o = \frac{1}{12} \left(\frac{R\gamma}{C} \right)^2 \left(1 + \frac{2V_o}{R\gamma} \right) \left(1 - e^{-\frac{6C}{R\gamma}\tau} \right) - \frac{R\gamma}{2C} \tau \tag{29a}$$

which satisfies the initial conditions $W = 0$ and $\dot{W} = V_o$ at $t = 0$ and

$$\tau_f = \frac{R\gamma}{6C} \log_e \left(1 + \frac{2V_o}{R\gamma} \right) \tag{29b}$$

is the time when $\dot{W} = 0$. Setting $u_o(\tau_f)$ for the first solution equal to $u_o(\tau_f)$ for the exact solution and solving for σ_o/σ_e one obtains

$$\frac{\sigma_o}{\sigma_e} = \frac{1}{\sqrt{3}} \left(\frac{R\gamma}{V_o} \right)^2 \left[\frac{2V_o}{R\gamma} - \log_e \left(1 + \frac{2V_o}{R\gamma} \right) \right] \tag{30}$$

therefore, the dominant solution may be taken as equation (24a) with (24b) and (30) used with it.

4.2 Perturbation Equation

Using equations (15a-c) equation (12b) becomes

$$a_{mn'\tau\tau} - \beta^2(\lambda_n - 2)c_{mn} + \frac{\bar{N}}{kh}\lambda_n a_{mn} + \alpha^2\beta^2[2(\lambda_n - 1) - 1](\lambda_n - 2)a_{mn'\tau} = 0 \qquad (31)$$

Solving equation (14a) for c_{mn} and substituting into equation (31) produces

$$a_{mn'\tau\tau} + \beta^2\tilde{\omega}_n a_{mn'\tau} + \frac{\bar{N}}{kh}\lambda_n a_{mn} = 0 \qquad (32a)$$

where $$\lambda_n = n(n + 1) \qquad (32b)$$

$$\tilde{\omega}_n = \frac{3(\lambda_n - 2)}{2\lambda_n - 1} + \alpha^2(2\lambda_n - 3)(\lambda_n - 2) \qquad (32c)$$

$$\beta^2 = \frac{C}{R\gamma^*} = \left(\frac{1}{2u_{o'\tau}} + \frac{C}{R\gamma}\right). \qquad (32d)$$

With a change of variable equation (32a) becomes

$$a_{mn'\xi\xi} - \tau_f\,\tilde{\omega}_n\left(\frac{C}{2V_o\xi} + \frac{C}{R\gamma}\right)a_{mn'\xi} - \tau_f^2\left(\frac{3}{2} + \frac{3V_o}{R\gamma}\xi\right)\lambda_n a_{mn} = 0 \qquad (33a)$$

where $$\xi = 1 - \tau/\tau_f \qquad (33b)$$

$$\frac{\bar{N}}{kh} = -\left(\frac{3}{2} + \frac{3V_o}{R\gamma}\,\xi\right) \tag{33c}$$

and
$$u_{o,\tau} = \frac{V_o}{C}\,\xi. \tag{33d}$$

Putting equation (33a) into a format similar to Florence [8], one gets

$$a_{mn,\xi\xi} - \left(Q_n^{(1)} + \frac{Q_n^{(2)}}{\xi}\right) a_{mn,\xi} - R_n^2\, a_{mn} = 0 \tag{34a}$$

where
$$Q_n^{(1)} = \tau_f\,\frac{C}{R\gamma}\,\tilde{\omega}_n \tag{34b}$$

$$Q_n^{(2)} = \tau_f\,\frac{C}{2V_o}\,\tilde{\omega}_n \tag{34c}$$

and
$$R_n^2 = \tau_f^2 \left(\frac{3}{2} + \frac{3V_o}{R\gamma}\,\xi\right) \lambda_n. \tag{34d}$$

As R_n^2 is still a function of time (ξ), in order to solve the equation using Florence's solution, the value of $\left(\frac{3}{2} + \frac{3V_o}{R\gamma}\,\xi\right)$ was set as a constant with respect to ξ. A value of 3/2 was used as we are primarily interested near $\xi = 0$ which negates the influence of the second term, thus

$$R_n^2 \equiv 3/2\ \tau_f^2\ \lambda_n \tag{34e}$$

As given in Florence [8] the solution to equation (34a) is

$$a_{mn}(\xi) = A_{mn}(\xi)\,\alpha_n + B_{mn}(\xi)\,\beta_n \tag{35}$$

$$\text{where} \quad A_{mn}(\xi) = \frac{\Gamma(1 + a_n + b_n)}{\Gamma(2 + b_n)} \left[\exp\{-(\Lambda_n + \Omega_n) + \Lambda_n\xi\}\right] (\Omega_n\xi)^{1+b_n} \left[-\{(\Lambda_n + 1 + b_n) U(\Omega_n) + \Omega_n U'(\Omega_n)\} M(\Omega_n\xi) + \{(\Lambda_n + 1 + b_n) M(\Omega_n) + \Omega_n M'(\Omega_n)\} U(\Omega_n\xi)\right] . \tag{36a}$$

$$\text{and} \quad B_{mn}(\xi) = \frac{\Gamma(1 + a_n + b_n)}{\Gamma(2 + b_n)} \left[\exp\{-(\Lambda_n + \Omega_n) + \Lambda_n\xi\}\right] (\Omega_n\xi)^{1+b_n} \left[U(\Omega_n) M(\Omega_n\xi) - M(\Omega_n) U(\Omega_n\xi)\right] 2\tau_f \tag{36b}$$

In equations (36a) and (36b), $M(\Omega_n\xi)$ and $U(\Omega_n\xi)$ are shorthand notations to represent Kummer functions [15-17] (confuent hypergeometric functions) of the first and second kind, respectively, that is

$$M(\Omega_n\xi) \equiv M(1 + a_n + b_n, 2 + b_n, \Omega_n\xi) \tag{37a}$$

$$U(\Omega_n\xi) \equiv U(1 + a_n + b_n, 2 + b_n, \Omega_n\xi) . \tag{37b}$$

Similarly

$$M(\Omega_n) \equiv M(1 + a_n + b_n, 2 + b_n, \Omega_n) \tag{37c}$$

$$U(\Omega_n) \equiv U(1 + a_n + b_n, 2 + b_n, \Omega_n) \tag{37d}$$

and primes over $M(\Omega_n)$ and $U(\Omega_n)$ denote differential coefficients with respect to $\Omega_n\xi$ evaluated at $\xi = 1$. The remaining symbols are

$$a_n = \frac{\Omega_n^{(2)}}{2}\left[\frac{\Omega_n^{(1)}}{\Omega_n} - 1\right] , \qquad b_n = \Omega_n^{(2)} \qquad (37e\text{-}f)$$

$$\Omega_n = \left(\Omega_n^{(1)\,2} + 4R_n^2\right)^{1/2} , \qquad \Lambda_n = \frac{\Omega_n^{(1)} - \Omega_n}{2} . \qquad (37g\text{-}h)$$

Chapter 5

Results and Discussion

Unfortunately, no experimental investigations appear to have been published on the dynamic instability of complete spherical shells and until reciently no theoretical work had been published. The results reported herein will then be compared with previous experimental investigations into the behavior of impulsively loaded cylindrical shells and in theory with the publications now in press. Equation (5.2) from Appendix E predicts mode numbers which are 3.173/3.624 = 0.876 times as large as Jones and Ahn's [13] estimates for similar aluminum spherical shells and are $3.173/74^{\frac{1}{4}}$ = 1.08 times larger than Florence and Vaughan's [4] estimates for similar aluminum cylindrical shells. However, as shown in Figure 2, a plot of the first five sets of data from Table 1, the equation for the critical harmonic is

$$n = 2.0(R/h)^{.605} \tag{38}$$

which compares favorably with Lyons [6] curve with a slope of 0.625.

R/h	Critical Harmonic n
10	8
18.25	12
30	15
50	21
100	32

Table 1: A Comparison of the Critical Mode Numbers for a Visco-plastic Spherical Shell with Differing R/h values, with R = 1.460 in., V_o = 9357.6 in/sec, γ = 6500 sec^{-1} σ_o = 46,000 psi and ρ = 0.0002523 $\text{lb-sec}^2/\text{in.}^4$.

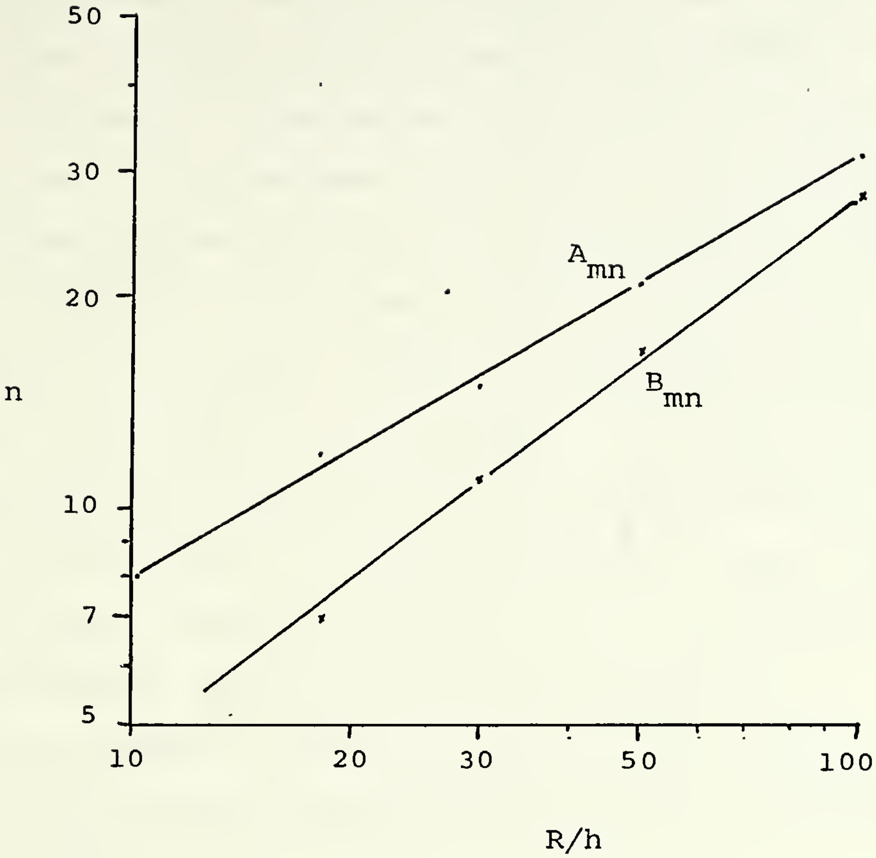
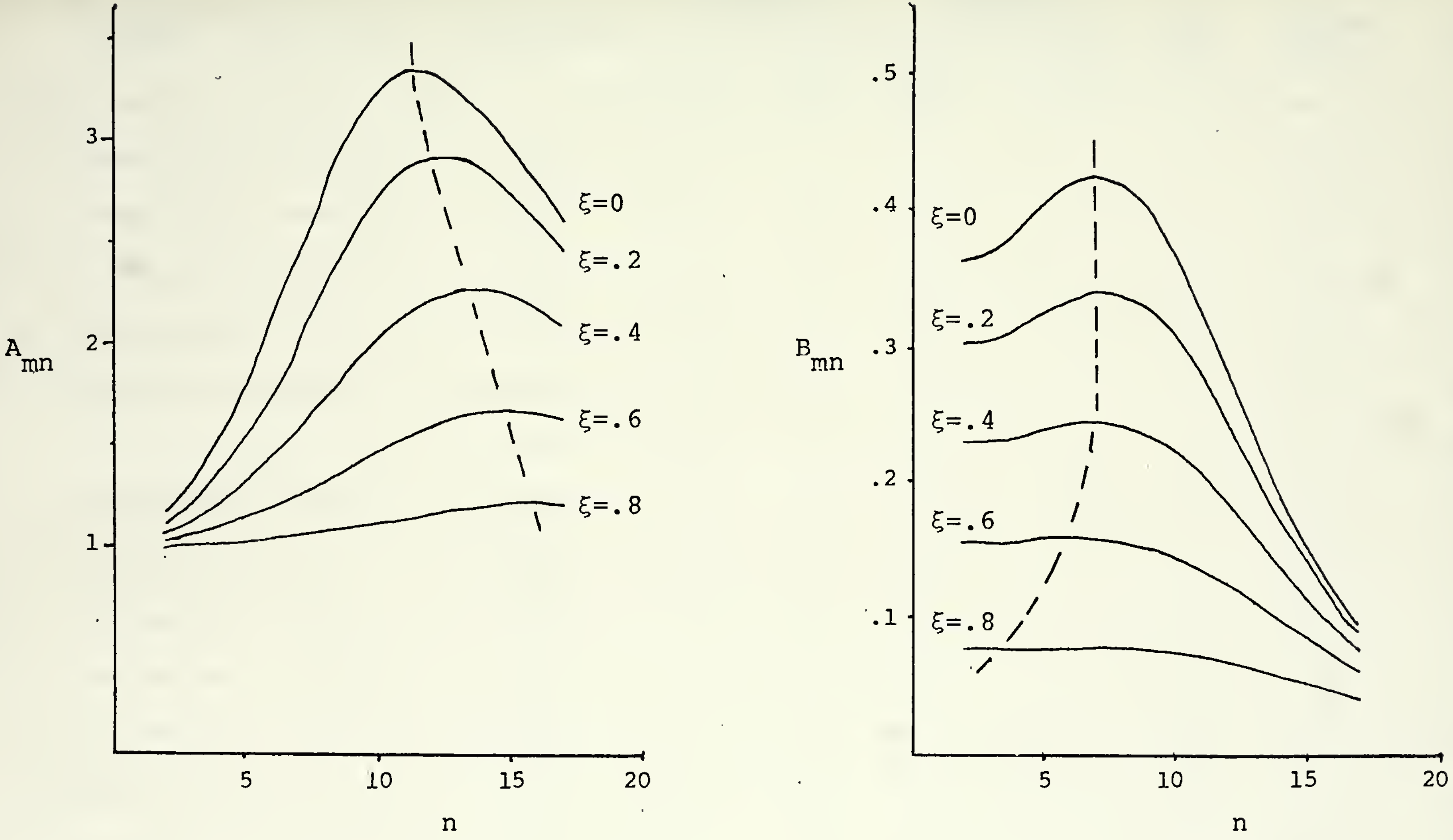


Figure 2: Critical Mode of Buckling as a Function of R/h with V_o = 9357.58 in/sec and γ = 6500 sec^{-1}.

Figure 3 shows the growth of the amplitude function with regard to time for the continuous spectrum of the mode numbers n. Showing addition figures later, this growth is faster for larger V_o, larger γ and for later stages of the process (small ξ). The change of preferred buckling mode is shown by a dashed line. It is evident from Table 2a that equation (38) provides a reasonable estimate of the preferred mode of the amplification function which is obtained numerically from the corresponding theoretical analysis using Kummer's functions and numerical integration of the equations. The use of the Kummer's functions for solution ran into problems because of the size of the functions and that differences of these very large numbers were removing the first five or six significant figures. Even double precision was not eliminating the problem, for the asymptotic expansion [16] is a truncating series and even with double precision can't give the necessary significant figures.

The Florence method gave good results that correlated well, up to four significant figures, with the numerical method. The problem with the Florence method occurred when the values of inputs to the Kummer function became so large i.e. greater than 8 that one has to use an asymptotic expansion. Also the last couple of computations with values just less than 8 one loses some accuracy. As was observed by Florence [8] and Wojewodzki [11] the critical modes of viscoplastic cylinders were constantly lower than modes of rigid-plastic cylinders. Likewise as shown from Table 2a,



(a)

(b)

Figure 3: Amplification Function at $\tau = .2, .4, .6, .8$ and $1.0\ \tau_f$ for (a) Displacement and (b) Velocity Imperfections for a Viscoplastic Spherical Shell with $R = 1.460$ in., $R/h = 18.25$, and $\gamma = 6500\ \text{sec}^{-1}$.

	Material	R(in)	R/h	V_o (in/sec)	Florence's Method		Numerical Integration		EQ. (38)	Expt. Cylinder	Ref.
					$A_{mn}(0)$	$B_{mn}(0)$	$A_{mn}(0)$	$B_{mn}(0)$			
1.	Steel 1015	1.494	10.4	5040.0	8	-	8	-	-	11	[1]
2.	Steel 1015	1.495	16.8	6000.0	11	5	10	-	-	15	[1]
3.	Steel 1015	1.494	22.6	5880.0	12	6	12	5	-	16	[1]
4.	Steel 1015	1.493	35.6	4560.0	16	8	15	6	-	28	[1]
5.	AL 2024 T3	1.495	9.9	8040.0	8	4	8	-	-	12	[1]
6.	AL 2024 T3	1.495	15.0	7080.0	10	5	10	4	-	12	[1]
7.	AL 2024 T3	1.495	19.4	9360.0	12	8	12	7	-	16	[1]
8.	AL 6061 T6	1.487	9.9	6480.0	8	-	8	-	8	9	[1]
9.	AL 6061 T6	1.488	14.7	6840.0	10	6	10	5	10	15	[1]
10.	AL 6061 T6	1.488	14.7	7440.0	10	6	10	5	10	16	[1]
11.	AL 6061 T6	1.490	19.6	7800.0	12	8	12	7	12	18	[1]
12.	AL 6061 T6	1.490	19.6	7680.0	12	8	12	7	12	18	[1]
13.	AL 6061 T6	1.453	15.3	4175.8	10	-	10	-	10	11-16	[5]
14.	AL 6061 T6	1.459	17.6	4571.7	11	5	11	-	11	14-15	[5]

Table 2a: A Comparison of the Critical Mode Numbers for a Viscoplastic Spherical Shell with the Corresponding Experimental Results on Cylindrical Shells. (Continued)

	Material	R(in)	R/h	V_o (in/sec)	Florence's Method		Numerical Integration		EQ. (38)	Expt. Cylinder	Ref.
					$A_{mn}(0)$	$B_{mn}(0)$	$A_{mn}(0)$	$B_{mn}(0)$			
15.	AL 6061 T6	1.468	22.6	5572.4	13	7	12	6	13	19-20	[5]
16.	AL 6061 T6	1.468	22.6	5749.3	13	8	12	6	13	15-16	[5]
17.	AL 6061 T6	1.495	9.9	6469.0	8	-	8	-	8	9	[6]
18.	AL 6061 T6	1.485	14.7	6993.4	10	6	10	5	10	15	[6]
19.	AL 6061 T6	1.485	14.7	7505.1	10	6	10	5	10	16	[6]
20.	AL 6061 T6	1.460	18.25	4678.8	11	5	11	-	12	-	[6]
21.	AL 6061 T6	1.460	18.25	5398.6	11	6	11	4	12	-	[6]
22.	AL 6061 T6	1.460	18.25	5758.5	11	6	11	5	12	14	[6]
23.	AL 6061 T6	1.460	18.25	9357.6	12	7	12	7	12	15	[6]
24.	AL 6061 T6	1.490	19.6	8252.5	12	8	12	7	12	18	[6]
25.	AL 6061 T6	1.490	19.6	7798 .2	12	8	12	7	12	18	[6]
26.	AL 6061 T6	1.464	20.92	8127.0	13	9	12	8	13	20-21	[6]
27.	AL 6061 T6	1.475	29.5	5522.7	15	9	14	8	15	23	[6]
28.	AL 6061 T6	1.475	29.5	7133.5	15	11	15	9	15	25	[6]

Table 2a: A Comparison of the Critical Mode Numbers for a Viscoplastic Spherical Shell with the Corresponding Experimental Results on Cylindrical Shells. (Continued)

	Material	R(in)	R/h	V_o(in/sec)	Florence's Method		Numerical Integration		EQ. (38)	Expt. Cylinder	Ref.
					$A_{mn}(0)$	$B_{mn}(0)$	$A_{mn}(0)$	$B_{mn}(0)$			
29.	AL 6061 T6	1.475	29.5	9204.6	15	11	15	11	15	24	[6]
30.	Steel 1015	1.510	10.5	2089.0	8	-	8	-	-	9	[7]
31.	Steel 1015	1.510	10.5	2411.0	8	-	8	-	-	9	[7]
32.	Steel 1015	1.510	10.5	2893.0	8	-	8	-	-	9	[7]
33.	Steel 1015	1.510	10.5	4581.0	8	-	8	-	-	9	[7]
34.	Steel 1015	1.488	16.7	3251.0	10	-	10	-	-	14	[7]
35.	Steel 1015	1.488	16.7	3381.0	10	-	10	-	-	14	[7]
36.	Steel 1015	1.488	16.7	4551.0	11	5	11	5	-	13	[7]
37.	Steel 1015	1.502	22.8	1921.0	12	-	12	-	-	22	[7]
38.	Steel 1015	1.502	22.8	2241.0	12	-	12	-	-	19-22	[7]
39.	Steel 1015	1.502	22.8	4162.0	13	7	13	7	-	19	[7]
40.	Steel 1015	1.494	35.6	2875.0	16	8	16	8	-	26	[7]
41.	Steel 1051	1.494	35.6	3354.0	16	9	16	9	-	26	[7]

Table 2a: A Comparison of the Critical Mode Numbers for a Viscoplastic Spherical Shell with the Corresponding Experimental Results on Cylindrical Shells.

Material	Ref.	σ_o (ksi)	ρ (lb-sec^2/in.^{4})	γ sec^{-1}
Steel 1015	[1]	102	.0007324	40.4
AL 2024 T3	[1]	56	.0002523	6500
AL 6061 T6	[1]	44	.0002523	6500
AL 6061 T6	[5]	44	.0002525	6500
AL 6061 T6	[6]	46	.0002523	6500
Steel 1015	[7]	30	.0007324	40.4

Table 2b: Values of σ_o, ρ and γ that Correspond to Materials in Table 2a.

Material No.	τ_f EQ. (24b)	C(10^3in/sec) EQ. (11d)
1	.094	16.1
2	.110	16.3
3	.108	16.3
4	.085	16.0
5	.158	15.4
6	.140	15.3
7	.182	15.5
8	.146	13.5
9	.153	13.5
10	.166	13.6
11	.173	13.6
12	.171	13.6
13	.097	13.2
14	.105	13.2
15	.127	13.4
16	.130	13.4
17	.142	13.8
18	.153	13.8
19	.164	13.9
20	.105	13.6

Table 2c: Values of τ_f and C that Correspond to Materials in Table 2a. (Continued)

Material No.	τ_f EQ.(24b)	$C(10^3$ in/sec) EQ.(11d)
21	.120	13.7
22	.128	13.7
23	.200	14.1
24	.179	14.0
25	.169	13.9
26	.179	14.0
27	.123	13.7
28	.156	13.9
29	.198	14.1
30	.118	8.5
31	.103	8.4
32	.145	8.6
33	.082	8.3
34	.071	8.2
35	.157	8.7
36	.119	8.5
37	.115	8.5
38	.158	8.7
39	.103	8.4
40	.087	8.3
41	.076	8.2

Table 2c: Values of $_f$ and C that Correspond to Materials in Table 2a.

the critical modes of spherical shells are considerably lower than those observed experimentally on cylindrical shells with the same material parameters and R/h ratios which were tested by the authors of references [1, 5, 6, 7], with the further observation that correlation seems to be closer at lower R/h values. Of course, n, in the present case, refers to the critical or most amplified harmonic* in the displacement field of a complete spherical shell which is expressed in terms of Legendre and associated Legendre polynomials (equation (15a)), while in reference [4] on cylindrical shells, it is associated with trignometric functions.

The concept of threshold or cirtical impulse has been used by numerous investigators to mark the smallest impulse that a structure can tolerate without excessive deformation. It is evident from Figure 4 which is a plot of Table 3 that the displacement amplification function is a highly non-linear function of the impulsive velocity (V_o). The critical velocity is very dependent on the value of γ used as shown in Figure 4.

It is evident from Figure 4 that the displacement functions at $\tau = \tau_f$ experience larger amplifications than do the velocity functions. As stated by Jones and Ahn [13], this suggests that shape imperfections might exercise a much more significant influence on the instability of spherical shells than variations in the initial velocity field.

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*The critical harmonics are always rounded off to the nearest integer.

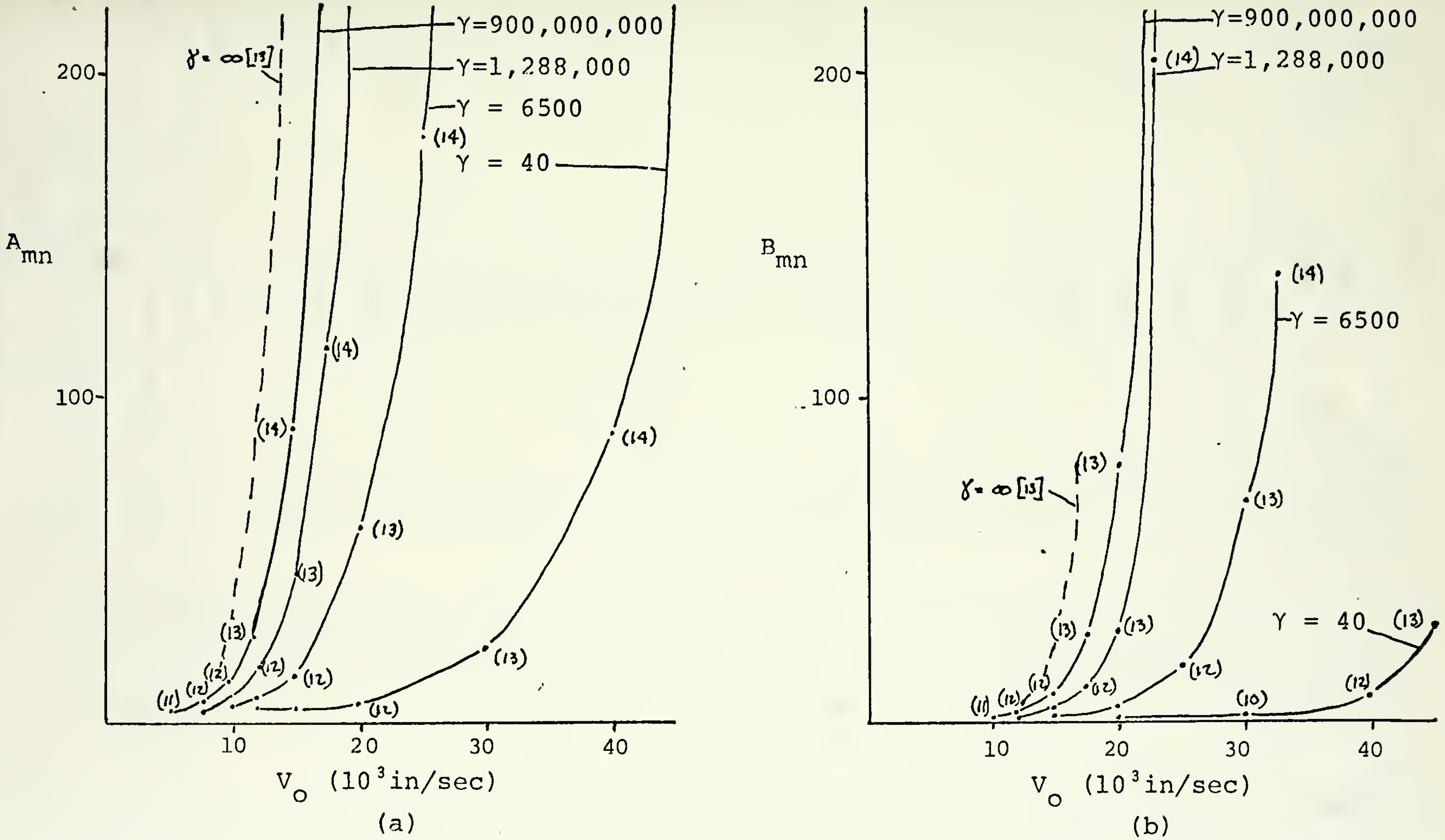

Figure 4: Variation of Largest Amplification Functions at $\tau = \tau_f$ with Initial Impulsive Velocity for (a) Displacement and (b) Velocity Imperfections for a Viscoplastic Spherical Shell with $R = 1.460$ in., $R/h = 18.25$ and Various Values of γ. The Numbers in Parentheses ( ) are the Critical Modes for $A_{mn}(0)$ and $B_{mn}(0)$.

V_o (10 in/sec)	γ $\sec^{-1}$	Critical Mode n	A_{mn}	Critical Mode n	B_{mn}
5	40	11	1.21	2	0.12
7.5	40	11	1.47	2	0.18
10	40	11	1.85	5	0.24
11	40	11	2.05	5	0.26
12	40	11	2.28	6	0.29
15	40	11	3.17	7	0.40
20	40	12	5.81	8	0.70
30	40	13	21.6	10	2.38
40	40	14	88.6	12	9.07
45	40	14	332.	13	28.6
5	6500	11	1.60	2	0.19
7.5	6500	11	2.51	6	0.31
10	6500	12	4.22	8	0.50
11	6500	12	5.28	8	0.61
12	6500	12	6.64	9	0.75
15	6500	12	13.6	10	1.45
20	6500	13	48.8	11	4.81
25	6500	14	190.	12	17.4
30	6500	15	788.	13	68.4
32.5	6500	15	1644.	14	138.

Table 3: Critical Mode Numbers and Displacement Function Amplitudes Versus Initial Velocities at Differing Values of γ with R = 1.460 in., R/h = 18.25, σ_o = 46,000 psi and ρ = 0.0002523 lb-sec^2/in.^{4}. (Cont.)

V_o (10 in/sec)	γ sec^{-1}	Critical Mode n	A_{mn}	Critical Mode n	B_{mn}
5	1,288,000	11	1.99	5	0.25
7.5	1,288,000	12	3.85	8	0.48
10	1,288,000	12	8.26	9	0.96
11	1,288,000	12	11.4	10	1.28
12	1,288,000	13	15.9	10	1.76
15	1,288,000	13	45.7	11	4.73
17.5	1,288,000	14	116.	12	11.4
20	1,288,000	14	304.	13	28.6
25	1,288,000	15	2334.	14	205.
5	900,000,000	11	2.24	6	0.29
7.5	900,000,000	12	4.89	8	0.61
10	900,000,000	12	11.7	10	1.35
11	900,000,000	13	17.5	10	1.98
12	900,000,000	13	25.7	11	2.81
15	900,000,000	14	89.4	12	9.19
17.5	900,000,000	14	274.	13	26.6
20	900,000,000	15	856.	13	80.3
22.5	900,000,000	15	2902.	14	252.

Table 3: Critical Mode Numbers and Displacement Function Amplitudes Versus Initial Velocities at Differing Values of γ with R = 1.460 in., R/h = 18.25, σ_o = 46,000 psi and = 0.0002523 lb-$\text{sec}^2/\text{in.}^4$.

It should be noted that the present theory is an approximate one since the linearized from of the constitutive equations for rate-sensitive material were used. In addition to the various simplifications and approximations which were introduced in the theoretical analysis, the influence of transverse shear stresses and deformations have been disregarded. Also, the influence of unloading has been neglected so that the effect of possible plastic behavior after $\tau = \tau_f$ has not been considered. Moreover, the influence of wave disturbance has been neglected. However, with final times (t_f) being in the neighborhood of 10-30 μsec, which can be calculated by using equation (11e) and Table 2c, then one should not neglect wave phenomena. As an example materials 1, 10, 20, 30, 40 have values of t_f = 8.7, 13.2, 11.3, 29.1, and 15.9 μsec, respectively.

Chapter 6

Conclusions

A theoretical investigation has been undertaken into the dynamic instability of complete spherical shells which are made from a viscoplastic material and loaded impulsively. Unfortunately, no experimental results are available to examine the validity, or otherwise, of the various theoretical predictions. However, the forms of the expressions for the critical mode number and threshold velocity are similar to those developed for cylindrical shells and spherical shells, except that the magnitude of the numerical coefficients are different. The threshold velocities for viscoplastic spherical shells came out higher than that for rigid plastic spherical shells with the same R/h ratios and material parameters. The viscoplastic theory predicts critical mode numbers for spherical shells which are similar to those observed experimentally on cylindrical shells with $10 < R/h < 30$ but the correlation was better at the lower range the critical mode numbers were only one to three less than the rigid-plastic prediction where as the higher range produced mode numbers of four to eight less than the rigid plastic prediction.

Chapter 7

Recommendations

To further the investigation of viscoplastic effects, the next logical step would be to look into the non-linear constitutive equations. This may result in a sufficient enough change so as to correct some of the problems of the threshold velocities.

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Appendix A

Constitutive Equations of Viscoplastic Spherical Shells

The constitutive equations for cylindrical shells which are made from a rigid viscoplastic material are developed in references [8], [10], [11], and [18]. This appendix contains a brief derivation of constitutive equations for viscoplastic spherical shells.

Consider a special linearized case of the constitutive equations for a rate-sensitive plastic material as derived by Perzyna [18]

$$\dot{\varepsilon}_{ij} = 2k\gamma F \frac{\partial F}{\partial \sigma_{ij}} \tag{1.1}$$

where $\dot{\varepsilon}_{ij}$ and σ_{ij} are components of the plastic strain rate and stress tensors, k is the yield stress in simple shear and γ is a physical constant of the material later termed the viscosity constant. γ is obtained by uniaxial tests as described later. In the above equation the function F, a function of strain rate and stress, is defined by $F = \sqrt{J_2}/k - 1$ where $J_2 = 1/2\, S_{ij}\, S_{ij}$ is the second invarient of the stress deviator with components S_{ij}. In accordance with the Levy-Mises flow rule for rigid perfectly plastic material, as $\gamma \to \infty$, $J_2 = k^2$ where $k = \sigma_o/\sqrt{3}$.

The results of uniaxial stress experiments produced the Cowper and Symonds [19] power-type relation

$$\sigma_e = \sigma_o \left[1 + \left(\frac{\dot{\varepsilon}_e}{\gamma} \right)^{1/\delta} \right] \tag{1.2}$$

which give the non-linearized form of function F. Inverting (1.2) we get

$$\dot{\varepsilon}_e = \gamma \left(\frac{\sigma_e}{\sigma_o} - 1 \right)^{\delta} \tag{1.3a}$$

where σ_e and $\dot{\varepsilon}_e$ are the equivalent stress $[\frac{3}{2} S_{ij} S_{ij}]^{\frac{1}{2}}$ and strain rate $[\frac{2}{3} \dot{\varepsilon}_{ij} \dot{\varepsilon}_{ij}]^{\frac{1}{2}}$ respectively and σ_o is the static uniaxial yield stress of the material and γ and δ are constants obtained experimentally for a given material

$$\sigma_e = [\sigma_{11}{}^2 - \sigma_{11} \sigma_{22} + \sigma_{22}{}^2 + 3\sigma_{12}{}^2]^{\frac{1}{2}} \tag{1.3b}$$

$$\dot{\varepsilon}_e = \frac{2}{\sqrt{3}} [\dot{\varepsilon}_{11}{}^2 + \dot{\varepsilon}_{11} \dot{\varepsilon}_{22} + \dot{\varepsilon}_{22}{}^2 + \dot{\varepsilon}_{12}{}^2]^{\frac{1}{2}}. \tag{1.3c}$$

Taking the linearized form of equation (1.2) we get

$$\sigma_e = \sigma_o \left(1 + \frac{\dot{\varepsilon}_e}{\gamma} \right) \tag{1.4}$$

where γ is redefine as explained in Appendix B.

In the case of plane stress ($\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$) equation (1.1) yields the following equations.

$$\dot{\varepsilon}_{11} = \frac{\gamma}{3} \left(\frac{\sqrt{J_2}}{k} - 1 \right) \frac{2\sigma_{11} - \sigma_{22}}{\sqrt{J_2}} \tag{1.5a}$$

$$\dot{\varepsilon}_{22} = \frac{\gamma}{3} \left(\frac{\sqrt{J_2}}{k} - 1 \right) \frac{2\sigma_{22} - \sigma_{11}}{\sqrt{J_2}} \tag{1.5b}$$

$$\dot{\varepsilon}_{12} + \dot{\varepsilon}_{21} = 2\gamma \left(\frac{\sqrt{J_2}}{k} - 1 \right) \frac{\sigma_{12}}{\sqrt{J_2}} \tag{1.5c}$$

where $$J_2 = \frac{1}{3}\sigma_e^{\,2}, \qquad k = \sigma_o/\sqrt{3}. \qquad (1.5d\text{-}e)$$

Inverting equations (1.5a-c) produces.

$$\sigma_{11} = \frac{k}{\gamma^*}\,(2\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22}) \qquad (1.6a)$$

$$\sigma_{22} = \frac{k}{\gamma^*}\,(2\dot{\varepsilon}_{22} + \dot{\varepsilon}_{11}) \qquad (1.6b)$$

$$\sigma_{12} = \frac{k}{\gamma^*}\left(\frac{\dot{\varepsilon}_{12} + \dot{\varepsilon}_{21}}{2}\right). \qquad (1.6c)$$

where $$\gamma^* = \gamma(1 - k/\sqrt{J_2}) = \gamma(1 - \sigma_o/\sigma_e). \qquad (1.6d)$$

Now, assume that the strain rates can be represented as follows:

$$\dot{\varepsilon}_{11} = \dot{e}_1 + z\,\dot{\kappa}_1 \qquad (1.7a)$$

$$\dot{\varepsilon}_{22} = \dot{e}_2 + z\,\dot{\kappa}_2 \qquad (1.7b)$$

$$\dot{\varepsilon}_{12} + \dot{\varepsilon}_{21} = \dot{e}_{12} + z\,\dot{\kappa}_{12}. \qquad (1.7c)$$

Considering that γ^* is not constant and in fact is a function of z, one assumes that the variation of γ^* with z is second order, a perturbation quantity. As shown in appendix B γ^* is dependent on $\dot{\varepsilon}_e$ and as shown in Figure 5, $\dot{\varepsilon}_1 = \dot{\bar{\varepsilon}}_1 + \dot{\varepsilon}_1'$ where $\dot{\varepsilon}_1'$ is a perturbation quantity dependent on z. Therefore, γ^* can be taken as a function of $\dot{\bar{\varepsilon}}_e$ and not a function of z. Now the constitutive equations can be integrated out as

$$N_1 = \int_{-h/2}^{h/2} \sigma_1 dz = \frac{kh}{\gamma^*}\,(2\dot{e}_1 + \dot{e}_2) \qquad (1.8a)$$

$$N_2 = \int_{-h/2}^{h/2} \sigma_2 dz = \frac{kh}{\gamma^*} (2\dot{e}_2 + \dot{e}_1) \tag{1.8b}$$

$$N_{12} = \int_{-h/2}^{h/2} \sigma_{12} dz = \frac{kh}{\gamma^*} \frac{\dot{e}_{12}}{2} \tag{1.8c}$$

$$M_1 = \int_{-h/2}^{h/2} \sigma_1 z dz = \frac{kh^3}{12\gamma^*} (2\dot{\kappa}_1 + \dot{\kappa}_2) \tag{1.8d}$$

$$M_2 = \int_{-h/2}^{h/2} \sigma_2 z dz = \frac{kh^3}{12\gamma^*} (2\dot{\kappa}_2 + \dot{\kappa}_1) \tag{1.8e}$$

$$M_{12} = \int_{-h/2}^{h/2} \sigma_{12} z dz = \frac{kh^3}{12\gamma^*} \frac{\dot{\kappa}_{12}}{2} \tag{1.8f}$$

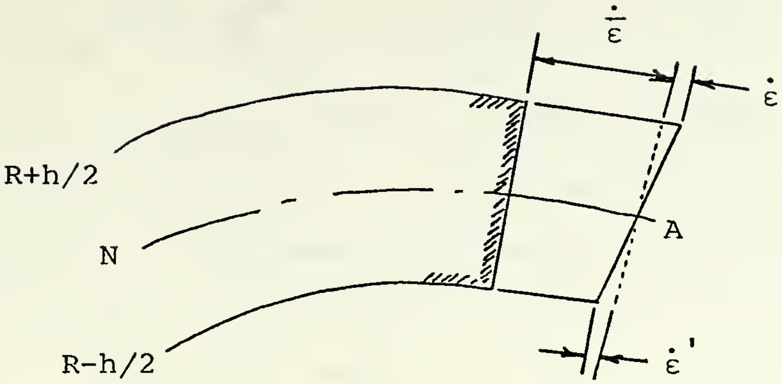

Figure 5: Variation of Strain Rate Across Thickness of Section.

Appendix B

Variations of γ and σ_o

Equation (1.4) from Appendix A is shown in figure 6 along with a straight line tangent to the Cowper and Symonds power type relation as suggested by Florence [7]. The equation Florence suggested is

$$\sigma_e = \sigma_{01} + \mu\dot{\varepsilon}_e \tag{2.1}$$

where μ is the slope of the tangent line, tangent at the maximum initial strain rate $\dot{\varepsilon}_e^*$, and σ_{01} is the intercept of this tangent line at static test strain rate. The slope of the tangent line is obtained by differentiations of equation (1.2) and evaluating the results at $\dot{\varepsilon}_e^*$

$$\left.\frac{d\sigma_e}{d\dot{\varepsilon}_e}\right|_{\dot{\varepsilon}_e = \dot{\varepsilon}_e^*} = \mu = \frac{\sigma_o}{\delta\gamma^{1/\delta}\,\dot{\varepsilon}_e^{*\,1-1/\delta}} \tag{2.2}$$

The value of σ_{01} is now obtained by equation equation (2.1) and (1.2) evaluated at $\dot{\varepsilon}_e^*$

$$\sigma_{01} = \sigma_o\left[1 + \left(\frac{\dot{\varepsilon}_e^*}{\gamma}\right)^{1/\delta}\left(1 - \frac{1}{\delta}\right)\right] \tag{2.3}$$

Putting equation (2.1) into the form of equation (1.4)

$$\sigma_e = \sigma_{01}\left(1 + \frac{\dot{\varepsilon}_e}{\sigma_{01}/\mu}\right) \tag{2.4}$$

where now σ_{01} and σ_{01}/μ compare with σ_o and γ from equation (1.4), respectively. Letting $\sigma_{01}/\mu = \gamma_1$

$$\gamma_1 = \delta\dot{\varepsilon}_e^* \left[\left(\frac{\gamma}{\dot{\varepsilon}_e^*}\right)^{1/\delta} + \left(1 - \frac{1}{\delta}\right)\right] \tag{2.5}$$

and

$$\sigma_e = \sigma_{01}\left(1 + \frac{\dot{\varepsilon}_e}{\gamma_1}\right). \tag{2.6}$$

A replacement of σ_o and γ with σ_{01} and γ_1 respectively in the equations would give the identical solution with the result being the use of the tangent line approximation.

The values of the material constants are $\delta = 5$ and $\gamma = 40.4 \text{ sec}^{-1}$ for hot rolled mild steel [20] and $\delta = 4$ and $\gamma = 6500 \text{ sec}^{-1}$ for 6061 T6 aluminum [21]. For thin spherical shells of radius R the initial strain rate is computed as $\dot{\varepsilon}_e^* = 2V_o/R$, where V_o is the uniform initial velocity.

From Chapter 2 equation (5b)

$$\gamma^* = \gamma(1 - \sigma_o/\sigma_e) \tag{2.7a}$$

where

$$k = \sigma_o/\sqrt{3} \quad \text{and} \quad \sqrt{J_2} = \sigma_e/\sqrt{3}. \tag{2.7b-c}$$

Solving equation (1.4) for σ_o/σ_e gives

$$\frac{\sigma_o}{\sigma_e} = \frac{\gamma}{\dot{\varepsilon}_e + \gamma}. \tag{2.8}$$

Since $1/\gamma^*$ is used more predominantly, substituting (2.8) into (2.7a) and inverting produces

$$\frac{1}{\gamma^*} = \frac{1}{\gamma} + \frac{1}{\dot{\varepsilon}_e}. \tag{2.9}$$

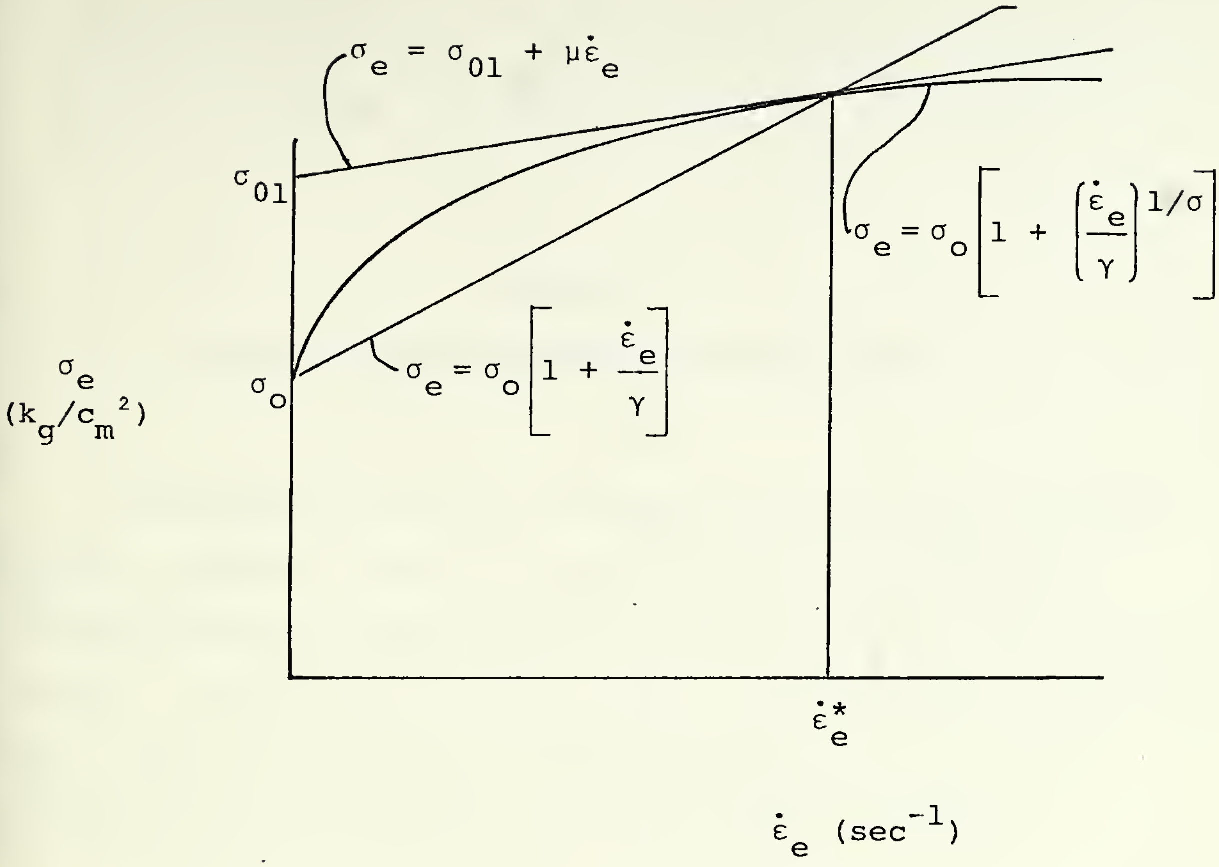

Figure 6: Cowper and Symonds Power Law and Linear Approximation to it [11].

Appendix C

Computer Program Using Florence Method

The following program was used to solve for $A_{mn}(\xi)$ and $B_{mn}(\xi)$, equations (36a) and (36b), by use of Mathematics Library Subroutine called HYPER which is a subroutine to solve Kummer's function of the first kind. Problems were encountered when taking the difference between functions as this was eliminating the first five or six significant figures.

```
C  MAIN PROGRAM USING FLORENCE METHOD OF SOLUTION                         FLOR0001
C     INPUT VALUES                                                        FLOR0002
C          RADIUS                        R                                FLOR0003
C          ASPECT RATIO (R/H)            ROH                              FLOR0004
C          INITIAL VELOCITY              VO                               FLOR0005
C          YIELD STRESS                  SIGMA                            FLOR0006
C          VISCOSITY COEF. (GAMMA)       GAM                              FLOR0007
C          DENSITY                       RHO                              FLOR0008
C          COWPER/SYMONDS POWER CONST    DELTA                            FLOR0009
C                                                                         FLOR0010
      DIMENSION AMN(40), BMN(40)                                          FLOR0011
      REAL *8  A, G, RM, CLDRM, DELTA                                     FLOR0012
      REAL *8 RMZ, RMO, RMP, URMZ, URMO, URMP, UMZ, UMO, UMP              FLOR0013
      INTEGER*4  N, IERR                                                  FLOR0014
   75 READ(5,100,END=99) R,ROH,VO, SIGMA, GAM, RHO, DELTA                 FLOR0015
      WRITE(6,101) R,ROH,VO, SIGMA, GAM, RHO, DELTA                       FLOR0016
   61 ASQ = 1./(12.*ROH**2)                                               FLOR0017
      RT3 = SQRT(3.0)                                                     FLOR0018
      SIGMA1 = SIGMA*(1.+(2./R*VO/GAM)**(1./DELTA)*(1.-1./DELTA))         FLOR0019
      GAM1 = DELTA*2./R*VO*((GAM/VO*R/2.)**(1./DELTA)+1.-1./DELTA)        FLOR0020
      C = SQRT(SIGMA1/RHO/RT3)                                            FLOR0021
      VND = R*GAM1/VO                                                     FLOR0022
      SOS = VND**2*(2./VND - ALOG(1.+2./VND))/RT3                         FLOR0023
      TAUF = VO*SOS/(2.*RT3*C)                                            FLOR0024
      WRITE(6,150) TAUF,SOS, C,SIGMA1,GAM1                                FLOR0025
C                                                                         FLOR0026
C  COMPUTATION OF ETA'S                                                   FLOR0027
C                                                                         FLOR0028
      DO 1 J=1,6                                                          FLOR0029
      ETA = 1. - 0.2*(J-1)                                                FLOR0030
      IND = 0                                                             FLOR0031
      YMAX = 0.                                                           FLOR0032
C                                                                         FLOR0033
C  COMPUTE VALUES OF AN(ETA) AND  BN(ETA)                                 FLOR0034
C                                                                         FLOR0035
      DO 2 M=2,40                                                         FLOR0036
```

```
      YN = M*(M+1)                                                       FLOR0037
      WN = 3.*(YN-2.)/(2.*YN-1.) + ASQ*(2.*YN-3.)*(YN-2.)                FLOR0038
      QN1 = TAUF*C*WN/(R*GAM1)                                           FLOR0039
      QN2 = TAUF*C*WN/(VO*2.)                                            FLOR0040
      RN2 = 1.5*TAUF**2*YN                                               FLOR0041
      ON = SQRT(QN1**2 + 4.*RN2)                                         FLOR0042
      ELN = (QN1-ON)/2.                                                  FLOR0043
      AN = QN2*(QN1/ON - 1.)/2.                                          FLOR0044
      BN = QN2                                                           FLOR0045
      AT = 1.+AN+BN                                                      FLOR0046
      GT = 2.+BN                                                         FLOR0047
      ZT = ON*ETA                                                        FLOR0048
      TEST = QN2*ALOG(QN1)                                               FLOR0049
      IF(TEST.GT.174.) GO TO 80                                          FLOR0050
      MAX = M                                                            FLOR0051
      DELTA = .000001                                                    FLOR0052
      IF (J.EQ.6) GO TO 51                                               FLOR0053
C                                                                        FLOR0054
C  COMPUTE M(A,G,Z)                                                      FLOR0055
C                                                                        FLOR0056
      A = AT                                                             FLOR0057
      G = GT                                                             FLOR0058
      Z = ZT                                                             FLOR0059
      IF(Z.GT.8.) GO TO 3                                                FLOR0060
      IF(G.GT.8.) GO TO 3                                                FLOR0061
      CALL HYPER(A,G,Z,RM,OLDRM,N,DELTA,IERR)                            FLOR0062
      GO TO 4                                                            FLOR0063
    3 CALL AHYP(A,G,Z,RM,OLDRM,N,DELTA,IERR)                             FLOR0064
    4 IF(IERR.EQ.0) GO TO 5                                              FLOR0065
    5 RMZ = RM                                                           FLOR0066
C                                                                        FLOR0067
C  COMPUTE  M(1+A-G, 2-G, Z)                                             FLOR0068
C                                                                        FLOR0069
      A = 1. + AT - GT                                                   FLOR0070
      G = 2. - GT                                                        FLOR0071
      Z = ZT                                                             FLOR0072
```

```
      IF(Z.GT.8.) GO TO 6                                               FLOR0073
      IF(G.GT.8.) GO TO 6                                               FLOR0074
      CALL HYPER(A,G,Z,RM,OLDRM,N,DELTA,IERR)                           FLOR0075
      GO TO 7                                                           FLOR0076
    6 CALL AHYP(A,G,Z,RM,OLDRM,N,DELTA,IERR)                            FLOR0077
    7 IF(IERR.EQ.0) GO TO 8                                             FLOR0078
    8 URMZ = RM                                                         FLOR0079
C                                                                       FLOR0080
C   COMPUTE  M(A,G, OMEGA N)                                            FLOR0081
C                                                                       FLOR0082
   51 A = AT                                                            FLOR0083
      G = GT                                                            FLOR0084
      Z = ON                                                            FLOR0085
      IF(Z.GT.8.) GO TO 9                                               FLOR0086
      IF(G.GT.8.) GO TO 9                                               FLOR0087
      CALL HYPER(A,G,Z,RM,OLDRM,N,DELTA,IERR)                           FLOR0088
      GO TO 10                                                          FLOR0089
    9 CALL AHYP(A,G,Z,RM,OLDRM,N,DELTA,IERR)                            FLOR0090
   10 IF(IERR.EQ.0) GO TO 11                                            FLOR0091
   11 RMO= RM                                                           FLOR0092
C                                                                       FLOR0093
C   COMPUTE  M'(A, G, OMEGA N)= A/G*M(A+1,G+1,ON)                       FLOR0094
C                                                                       FLOR0095
      A = AT+1.                                                         FLOR0096
      G = GT+1.                                                         FLOR0097
      Z = ON                                                            FLOR0098
      IF(Z.GT.8.) GO TO 12                                              FLOR0099
      IF(G.GT.8.) GO TO 12                                              FLOR0100
      CALL HYPER(A,G,Z,RM,OLDRM,N,DELTA,IERR)                           FLOR0101
      GO TO 13                                                          FLOR0102
   12 CALL AHYP(A,G,Z,RM,OLDRM,N,DELTA,IERR)                            FLOR0103
   13 IF(IERR.EQ.0) GO TO 14                                            FLOR0104
   14 RMP = RM*AT/GT                                                    FLOR0105
      IF(J.NE.6) GO TO 200                                              FLOR0106
      AMN(M) = EXP(-(ELN+ON))*((ELN+1.+BN)*RMO+ON*RMP)/(1.+BN)          FLOR0107
      BMN(M) = EXP(-(ELN+ON))*RMO/(1.+BN)*2.*TAUF                       FLOR0108
```

```
      GO TO 68                                                          FLOR0109
C                                                                       FLOR0110
C  COMPUTE  M(1+A-G, 2-G, OMEGA N)                                      FLOR0111
C                                                                       FLOR0112
  200 A = 1.+AT-GT                                                      FLOR0113
      G = 2.-GT                                                         FLOR0114
      Z = CN                                                            FLOR0115
      IF(Z.GT.8.) GO TO 15                                              FLOR0116
      IF(G.GT.8.) GO TO 15                                              FLOR0117
      CALL HYPER(A,G,Z,RM,OLDRM,N,DELTA,IERR)                           FLOR0118
      GO TO 16                                                          FLOR0119
   15 CALL AHYP(A,G,Z,RM,CLDRM,N,DELTA,IERR)                            FLOR0120
   16 IF(IERR.EQ.0) GO TO 17                                            FLOR0121
   17 URMC = RM                                                         FLOR0122
C                                                                       FLOR0123
C  COMPUTE M(1+A-G,1-G, OMEGA N)                                        FLOR0124
C                                                                       FLOR0125
      A = 1.+AT-GT                                                      FLOR0126
      G = 1.-GT                                                         FLOR0127
      Z = CN                                                            FLOR0128
      IF(Z.GT.8.) GO TO 18                                              FLOR0129
      IF(G.GT.8.) GO TO 18                                              FLOR0130
      CALL HYPER(A,G,Z,RM,OLDRM,N,DELTA,IERR)                           FLOR0131
      GO TO 19                                                          FLOR0132
   18 CALL AHYP(A,G,Z,RM,CLDRM,N,DELTA,IERR)                            FLOR0133
   19 IF(IERR.EQ.0) GO TO 20                                            FLOR0134
   20 URMP = RM                                                         FLOR0135
      X = 1.+AT-GT                                                      FLOR0136
      CALL NEGAM(X,GM)                                                  FLOR0137
      G1 = GM                                                           FLOR0138
      X = GT                                                            FLOR0139
      CALL NEGAM(X,GM)                                                  FLOR0140
      G2 = GM                                                           FLOR0141
      X = AT                                                            FLOR0142
      CALL NEGAM(X,GM)                                                  FLOR0143
      G3 = GM                                                           FLOR0144
```

```
      X = 2.-GT                                                         FLOR0145
      CALL NEGAM(X,GM)                                                  FLOR0146
      G4 = GM                                                           FLOR0147
      X = GT+1.                                                         FLOR0148
      CALL NEGAM(X,GM)                                                  FLOR0149
      G5 = GM                                                           FLOR0150
      X = AT+1.                                                         FLOR0151
      CALL NEGAM(X,GM)                                                  FLOR0152
      G6 = GM                                                           FLOR0153
      X = 1.-GT                                                         FLOR0154
      CALL NEGAM(X,GM)                                                  FLOR0155
      G7 = GM                                                           FLOR0156
      PI = 3.14159                                                      FLOR0157
C                                                                       FLOR0158
C  COMPUTE U(A,G,Z)                                                     FLOR0159
C                                                                       FLOR0160
      UMZ = PI/SIN(PI*GT)*(RMZ/ G1/G2  - ZT**(1.-GT)/G3*URMZ/G4)        FLOR0161
C                                                                       FLOR0162
C  COMPUTE  U(A,G,OMEGA N)                                              FLOR0163
C                                                                       FLOR0164
      UMO = PI/SIN(PI*GT)*(RMO/ G1/G2  - CN**(1.-GT)/G3*URMO/G4)        FLOR0165
C                                                                       FLOR0166
C  COMPUTE  U'(A,G,OMEGA N)                                             FLOR0167
C                                                                       FLOR0168
      USMP = RMP/AT/G1*GT/G5 - URMP/ON**GT/G6/G7                        FLOR0169
      UMP = -AT*PI/SIN(PI*(GT+1.))*USMP                                 FLOR0170
C                                                                       FLOR0171
C  COMPUTE AN(ETA)                                                      FLOR0172
C                                                                       FLOR0173
      VAL = G3/G2*EXP(-(ELN+ON)+ELN*ETA)*ZT**(1.+BN)                    FLOR0174
      F = ELN+1.+BN                                                     FLOR0175
      AMN(M)=VAL*(-(F*UMO+ON*UMP)*RMZ+(F*RMO+CN*RMP)*UMZ)               FLOR0176
C                                                                       FLOR0177
C  COMPUTE  BN(ETA)                                                     FLOR0178
C                                                                       FLOR0179
      BMN(M)=VAL*(UMO*RMZ - RMO*UMZ)*2.*TAUF*(-1.)                      FLOR0180
```

```
 68 IF(YMAX.GT.AMN(M)) GO TO 69                                                       FLOR0181
    YMAX = AMN(M)                                                                     FLOR0182
    GO TO 2                                                                           FLOR0183
 69 IND = IND+1                                                                       FLOR0184
    IF(IND.EQ.6) GO TO 70                                                             FLOR0185
  2 CONTINUE                                                                          FLOR0186
 70 MAX = M                                                                           FLOR0187
 80 CONTINUE                                                                          FLOR0188
    WRITE(6,103) ETA                                                                  FLOR0189
    WRITE(6,104) (I,AMN(I),BMN(I),I=2,MAX)                                            FLOR0190
  1 CONTINUE                                                                          FLOR0191
    GO TO 75                                                                          FLOR0192
 99 STOP                                                                              FLOR0193
100 FORMAT(7F10.3)                                                                    FLOR0194
101 FORMAT('1',4X,'R=',F12.3,'  IN.', 8X,'R/H=',F9.1,15X,'VEL=',F10.3,                FLOR0195
   1'  IN/SEC',//,5X,'SIGMA=',F8.1,'  PSI', 8X,'GAMMA=',F7.1, ' SEC-1'                FLOR0196
   2,8X,'RHO=',F10.8,'  LBS-SEC**2/IN**4',//,5X,'DELTA=',F8.1)                        FLOR0197
103 FORMAT('0',4X,'ETA=',F10.4,6X,'N=',5X,'AN(ETA)=',20X,'BN(ETA)=')                  FLOR0198
104 FORMAT(' ',24X,I2,5X,E16.7,7X,E16.7)                                              FLOR0199
150 FORMAT('0    TAUF=',E17.7,5X,'SOS=',E16.7,7X,'C=',E16.7,//,5X,                    FLOR0200
   1'SIGMA1=',E15.7,'     GAM1=',E15.7)                                               FLOR0201
    END                                                                               FLOR0202
```

```
C  SUBROUTINE TO COMPUTE KUMMER FUNC IN ASYMPTOTIC EXPANSION                AHYP0001
C                                                                           AHYP0002
      SUBROUTINE AHYP(A, G, Z, RM, OLDRM, N, DELTA, IERR)                   AHYP0003
      REAL *8  A, G, RM, OLDRM, DELTA,TERM,TERM1,OVL,VL                     AHYP0004
      IERR = 0                                                              AHYP0005
      AS = 1.-A                                                             AHYP0006
      GS = G-A                                                              AHYP0007
      VL = 1.                                                               AHYP0008
      TERM = 1.                                                             AHYP0009
      AT2 = 1000000.                                                        AHYP0010
      DO 5 N=1,130                                                          AHYP0011
      TERM1 = (AS+N-1.)*(GS+N-1.)/(N*Z)                                     AHYP0012
    6 TERM = TERM*TERM1                                                     AHYP0013
      AT1 =DABS(TERM)                                                       AHYP0014
      AB = AS+N-1.                                                          AHYP0015
      GB = GS+N-1.                                                          AHYP0016
C                                                                           AHYP0017
C  TEST IF AB OR GB ARE NEGATIVE                                            AHYP0018
C                                                                           AHYP0019
      IF(AB.LT.0.) GO TO 4                                                  AHYP0020
      IF(GB.LT.0.) GO TO 4                                                  AHYP0021
C                                                                           AHYP0022
C  TEST IF ABSOLUTE VALUE OF NEW TERM IS GREATER THAN LAST TERM             AHYP0023
C                                                                           AHYP0024
      IF(AT1.GT.AT2) GO TO 7                                                AHYP0025
    4 AT2 = AT1                                                             AHYP0026
      OVL = VL                                                              AHYP0027
      VL = OVL + TERM                                                       AHYP0028
      ERR =DABS(1. - OVL/VL)                                                AHYP0029
      IF(ERR.LE.DELTA) GO TO 8                                              AHYP0030
    5 CONTINUE                                                              AHYP0031
      GO TO 8                                                               AHYP0032
    7 N = N-1                                                               AHYP0033
      IERR = 3                                                              AHYP0034
    8 IF(N.EQ.130) IERR = 5                                                 AHYP0035
      X = G                                                                 AHYP0036
```

```
      CALL NEGAM(X,GM)                                                  AHYP0037
      GG = GM                                                           AHYP0038
      X = A                                                             AHYP0039
      CALL NEGAM(X,GM)                                                  AHYP0040
      GA = GM                                                           AHYP0041
      T1 = GG/GA*Z**(A-G)*EXP(Z)                                        AHYP0042
      OLDRM = T1*CVL                                                    AHYP0043
      RM = T1*VL                                                        AHYP0044
      RETURN                                                            AHYP0045
      END                                                               AHYP0046
```

```
C  SUBROUTINE TO COMPUTE GAMMA FUNC WITH POSSIBLE NEGATIVE VALUES OF ARG.      NGAM0001
C                                                                              NGAM0002
      SUBROUTINE NEGAM(X,GM)                                                   NGAM0003
      GA = 0.                                                                  NGAM0004
      XT = -X                                                                  NGAM0005
      IF(XT.GT.55) GO TO 200                                                   NGAM0006
      DO 5 J=1,50                                                              NGAM0007
      IF(XT.LT.0.) GO TO 100                                                   NGAM0008
      GA = GA - ALOG(XT)                                                       NGAM0009
      XT = XT - 1.                                                             NGAM0010
    5 CONTINUE                                                                 NGAM0011
  100 N = J - 1                                                                NGAM0012
      GN = ALGAMA(-XT) + GA                                                    NGAM0013
      GM = (-1)**N*EXP(GN)                                                     NGAM0014
      RETURN                                                                   NGAM0015
  200 GM = 0.                                                                  NGAM0016
      RETURN                                                                   NGAM0017
      END                                                                      NGAM0018
```

Appendix D

Computer Program Using Numerical Method

The following program solved the basic equation (34a) for $A_{mn}(\xi)$ and $B_{mn}(\xi)$ by use of Mathematics Library Subroutine called DQSYK which is a subroutine to solve up to fourth order simultaneous differential equations with initial conditions. This program used equation (34d).

```
C  PROGRAM TC SCLVE SECCND CRDER DIFFERENTIAL EQUATION
C     INPUT VALUES
C          RADIUS                        R
C          ASPECT RATIO (R/H)            ROH
C          INITIAL VELCCITY              VO
C          YIELD STRESS                  SIGMA
C          VISCCSITY COEF. (GAMMA)       GAM
C          DENSITY                       RHO
C          CONSTANT (FLCRENCE SOLN) /    CCNST
C          COWPER/SYMONDS POWER          DELTA
C
      INTEGER*4 N,KDEG(1),IFLAG,MXSTEP,KSTEP,KERMAX,KORDER(1)
      REAL*4 EPS(1),HMIN,HMAX,ERRMAX,AMN(6,50),BMN(6,5C)
      REAL*8 T,Y(2),F(1),H,CELT,TFINAL,YSAVF(2),DTABLE(17,1)
      DATA N,KDEG(1),EPS(1),H,HMIN,HMAX,DELT,TFINAL,MXSTEP,KORDER(1)/1,
     *2,.C00C1,-.05,.C01,.1,-.2,.CC001,25,6/
   75 REAC(5,100,ENC=99) R,ROH,VO, SIGMA, GAM, RHO, CONST ,DELTA
      WRITE(6,101) R,RCH,VC, SIGMA, GAM, RHO, CONST,DELTA
   61 ASQ = 1./(12.*RCH**2)
      RT3 = SQRT(3.0)
      SIGMA1 = SIGMA*(1.+(2./R*VC/GAM)**(1./DELTA)*(1.-1./DELTA))
      GAM1 = DELTA*2./R*VC*((GAM/VO*R/2.)**(1./DELTA)+1.-1./DELTA)
      C = SQRT(SIGMA1/RHO/RT3)
      VNC = R*GAM1/VO
      SOS = VND**2*(2./VND - ALOG(1.+2./VND))/RT3
      TAUF = VO*SOS/(2.*RT3*C)
      WRITE(6,150) TAUF,SOS, C,SIGMA1,GAM1
      IND=0
      MAX = 1
      YMAX = 0.
      DO 2 M=2,40
      YN = M*(M+1)
      WN = 3.*(YN-2.)/(2.*YN-1.) + ASQ*(2.*YN-3.)*(YN-2.)
      QN1 = TAUF*C*WN/(R*GAM1)
      QN2 = TAUF*C*WN/(VO*2.)
      RN2 = TAUF**2*YN
```

```
      T = 1.                                                            NUM10037
      Y(1) = 1.                                                         NUM10038
      Y(2) = 0.                                                         NUM10039
      J = 0                                                             NUM10040
      CALL DQSYK(N,T,Y,F,KDEG,EPS,IFLAG,H,HMIN,HMAX,DELT,TFINAL,MXSTEP, NUM10041
     1KSTEP,KERMAX,ERRMAX,KORDER,YSAVE,DTABLE)                          NUM10042
      GO TO 20                                                          NUM10043
   10 CALL DQSYK1                                                       NUM10044
   20 GO TO(22,22,23,24,25,26,27,28),IFLAG                              NUM10045
   22 F(1) = (QN1 + QN2/T)*Y(2) + RN2*Y(1)*(1.5+3.*VO*T/R/GAM1)         NUM10046
      GO TO 10                                                          NUM10047
   23 J = J + 1                                                         NUM10048
      AMN(J,M) = Y(1)                                                   NUM10049
      GO TO 10                                                          NUM10050
   24 AMN(6,M) = Y(1)                                                   NUM10051
      IF(YMAX.GT.Y(1)) GO TO 31                                         NUM10052
      YMAX = Y(1)                                                       NUM10053
      GO TO 30                                                          NUM10054
   31 IND=IND+1                                                         NUM10055
      IF(IND.EQ.6) MAX=M                                                NUM10056
      GO TO 30                                                          NUM10057
   25 GO TO 10                                                          NUM10058
   26 EPS(1) = 32.*ERRMAX*EPS(1)                                        NUM10059
      GO TO 10                                                          NUM10060
   27 HMIN = DABS(H)                                                    NUM10061
      GO TO 10                                                          NUM10062
   28 WRITE(6,190)                                                      NUM10063
      STOP                                                              NUM10064
   30 CONTINUE                                                          NUM10065
      T = 1.                                                            NUM10066
      Y(1) = 0.                                                         NUM10067
      Y(2) =-2.*TAUF                                                    NUM10068
      J = 0                                                             NUM10069
      CALL DQSYK(N,T,Y,F,KDEG,EPS,IFLAG,H,HMIN,HMAX,DELT,TFINAL,MXSTEP, NUM10070
     1KSTEP,KERMAX,ERRMAX,KORDER,YSAVE,DTABLE)                          NUM10071
   35 CALL DQSYK1                                                       NUM10072
```

```
      GO TO 40                                                          NUM10073
   40 GO TO(42,42,43,44,45,46,47,48),IFLAG                              NUM10074
   42 F(1) = (QN1 + QN2/T)*Y(2) + RN2*Y(1)*(1.5+3.*VO*T/R/GAM1)         NUM10075
      GO TO 35                                                          NUM10076
   43 J = J + 1                                                         NUM10077
      BMN(J,M) = Y(1)                                                   NUM10078
      GO TO 35                                                          NUM10079
   44 BMN(6,M) = Y(1)                                                   NUM10080
      IF(MAX.EQ.M) GO TO 49                                             NUM10081
      GO TO 2                                                           NUM10082
   45 GO TO 35                                                          NUM10083
   46 EPS(1) = 32.*ERRMAX*EPS(1)                                        NUM10084
      GO TO 35                                                          NUM10085
   47 HMIN = DABS(H)                                                    NUM10086
      GO TO 35                                                          NUM10087
   48 WRITE(6,190)                                                      NUM10088
      STOP                                                              NUM10089
    2 CONTINUE                                                          NUM10090
   49 DO 50 KO=1,6                                                      NUM10091
      ETA = 1. - 0.2*(KO-1)                                             NUM10092
      WRITE(6,103) ETA                                                  NUM10093
      WRITE(6,176)(I,AMN(KO,I),BMN(KO,I),I=2,MAX)                       NUM10094
   50 CONTINUE                                                          NUM10095
      GO TO 75                                                          NUM10096
   99 STOP                                                              NUM10097
  100 FORMAT(8F10.3)                                                    NUM10098
  101 FORMAT('1',4X,'R=',F10.3,'  IN.',10X,'R/H=',F8.3,15X,'VEL=',F10.3,NUM10099
     1'  IN/SEC',//,5X,'SIGMA=',F7.0,'  PSI', 9X,'GAMMA=',F6.0,'  SEC-1'NUM10100
     2,8X,'RHO=',F10.8,'  LBS-SEC**2/IN**4',//,5X,'CONST=',F6.2,15X,    NUM10101
     3'DELTA=',F3.1)                                                    NUM10102
  103 FORMAT('0',4X,'ETA=',F10.4,6X,'N=',5X,'AN(ETA)=',20X,'BN(ETA)=')  NUM10103
  150 FORMAT('0    TAUF=',E16.7,'  SOS=',E16.7,'  C=',E16.7,//,5X,      NUM10104
     1'SIGMA1=',E16.7,'     GAM1=',E16.7)                               NUM10105
  176 FORMAT(' ',24X,I2,5X,E16.7,7X,E16.7)                              NUM10106
  190 FORMAT('0  IFLAG = 8')                                            NUM10107
      END                                                               NUM10108
```

Appendix E

Critical Impulse

It is straightforward to show when $\lambda_n >> 1$ that λ_n can be eliminated from equations (34c) and (34e) and using equation (24b) to give

$$\frac{\rho V_o^2}{\sigma_o} = \frac{8}{\sqrt{3}} \frac{R_n^2}{(\sigma_o/\sigma_e)^2} \frac{h/R}{\left\{\frac{24\sqrt{3}\, Q_n^{(2)}}{(\sigma_o/\sigma_e)} - 9\right\}^{1/2}} \qquad (5.1)$$

where σ_o/σ_e is defined in equation (30). As shown in Table 4 the values of σ_o/σ_e lie within the range $0.98 < \sigma_o/\sigma_e < 1.16$ with a medium value $\sigma_o/\sigma_e \simeq 1.07$. Also the values of $Q_n^{(2)}$ lie within the range $1.27 < Q_n^{(2)} < 4.42$ with a medium value of 2.84. It is straightforward to show when using equation (34c) with $n >> 1$, that the critical harmonic

$$n = 3.173(R/h)^{\frac{1}{2}} \qquad (5.2)$$

is associated with these values of σ_o/σ_e and $Q_n^{(2)}$.

The threshold of dynamic plastic instability could be said to occur when $A_{mn}(0) = X$, where X is an arbitrary positive number. If X = 100, then the initial displacement imperfections in the critical mode are magnified by a factor of one hundred. This amplification is achieved when $R_n \simeq 1.7$ according to Florence [3] asymptotic formula for equation (36a)

$$A_{mn}(0) = \exp\left[-(\Lambda_n + \Omega_n)\right] \left\{ \frac{(\Lambda_n+1+b_n)M(\Omega_n)+\Omega_n M'(\Omega_n)}{1 + b_n} \right\} \qquad (5.3)^*$$

where the right hand quantities are defined in equations (37a-h).

V_o (10^3in/sec)	γ (sec^{-1})	σ_o/σ_e	Critical Harmonic	$Q_n^{(2)}$
5	40	1.00	11	1.27
30	40	0.98	13	2.23
5	6500	1.06	11	1.35
30	6500	1.03	15	3.99
5	1,288,000	1.11	11	1.41
25	1,288,000	1.10	15	4.26
5	900,000,000	1.16	11	1.47
22.5	900,000,000	1.14	15	4.42

Table 4: Variation of σ_o/σ_e and $Q_n^{(2)}$ to various values of and γ with R = 1.460 in., R/h = 18.25 σ_o = 46,000 psi and ρ = 0.0002523 lb-sec^2/in.^{4}.

Finally equation (5.1) predicts a non-dimensionalized critical or threshold velocity

$$\frac{\rho V_o^{\,2}}{\sigma_o} = 1.11\ h/R \qquad (5.4)$$

---

[*] This asymptotic formula gave agreement with numerical values at ξ = 0 up to 4 significant figures.

This value does not compare favorably with Jones' [13] in which his coefficient is 17.8, since the velocities required should be higher than in the in rigid-plastic case.

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